Group Theory 8

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Theorem:- Let p be a prime and a group G of order p^n is cyclic if and only if it is an abelian group having a unique subgroup of order p.

Proof:- $a \in G$ and $Order(a) = n$, $n = mk \Rightarrow a^k$ has order m

$G = \langle a \rangle$, $S_1 = \langle a^{n/k} \rangle$ is a subgroup of order k

$Ord(a^{n/k}) = k$

$S_2 = \langle b \rangle$ of order k

$b^k = 1$ $\quad b = a^q \quad a^{qk} = 1$

$\langle b \rangle \subseteq \langle a^q \rangle \rightarrow q = n/k \quad \Rightarrow$ So unique for p in p^n cyclic group

Q) G is a non-cyclic group of order 121. How many subgroup G has.

Ans:- $\mathbb{Z}_{11} \times \mathbb{Z}_{11}$, 1, +12 subgroups from non-identity elements of $\mathbb{Z}_{11}$ (generated by)

$a \in \mathbb{Z}_{11}$, $Ord(a) = 11$ for $a \neq e$ $\quad \frac{120}{10} = 12$ subgroups

Normal Subgroups:-

If S and T are non-empty subsets of a group G, then

$$ST = \{st;\ s \in S,\ t \in T\}$$

If $S \leq G$, $t \in G$, $T = \{t\}$ then ST is the right coset St.

Product Formula :- If S and T are subgroups of a finite group,

$$|ST||S\cap T| = |S||T|$$

Proof :- $\varphi : S\times T \to ST$ $\quad \varphi$ is surjection
$\quad\quad s,t \to st$

We need to show that if $x\in ST$ then $|\varphi^{-1}(x)| = |S\cap T|$

$\varphi^{-1}(x) = (xt, t^{-1})$ $\quad \varphi^{-1}(x) = (sk, k^{-1}t)$
$\varphi^{-1}(x) = (s, s^{-1}x)$ $\quad st = x \quad k\in S\cap T$

$(s_1, t_1), (s_2, t_2) \in \varphi^{-1}(x)$

$s_1, s_2 \in S, t_1, t_2 \in T$
$st = x = s_2t_2 \quad \in S\cap T$

$\Rightarrow |\varphi^{-1}(x)| = |S\cap T|$ this is for each x

So we get $|ST|\,|S\cap T| = |S\times T| = |S||T|$
how many such x's

Definition :- A subgroup $K\leq G$ is a normal subgroup if $gKg^{-1} = K$ for every $g\in G$.

denoted by $K \trianglelefteq G$

Same as $g^{-1}Kg = K$

• The Kernel K of a homomorphism $f : G\to H$ is normal subgroup

K is Kernel $\quad f(gkg^{-1}) = f(g)f(k)f(g)^{-1} = f(g)f(g)^{-1} = e$

$gKg^{-1} = K$

K is kernel $\quad f(gkg^{-1}) = f(g)f(k)f(g)^{-1} = \ldots$

$$gKg^{-1} = K$$

•> If $x \in G$ then a conjugate of x in G is an element of the form axa^{-1} for some $a \in G$.

Q> If S is a subgroup of G, then $SS = S$. Prove it

Q> If S is a finite non-empty subset of G with $SS = S$ then is S a subgroup? $\Rightarrow$

Ans:- $s_1 s_2 \in S$ $\quad s_1^2 \in S \quad s_1^2 s_1 \in S \quad s_1^4 \in S \quad s_1^n = e$ as S is finite

So $s_1^{-1} \in S$

So S is a subgroup.

$s_1^{n+1} = s_1$

$s_1^n = e$

•> If S is infinite then S may not be a subgroup

$G = \mathbb{Z}$, $\quad S = \{1, 2, \ldots\} \Rightarrow SS = S \rightarrow$ not subgroup

Q> Let H and D be two subgroups of a group such that $H \not\subset D$ and $D \not\subset H$. Prove that $H \cup D$ is never a group.

Ans:- Let us suppose $H \cup D$ is a group.

Let $a \in H \setminus D$ and $b \in D \setminus H$

Then $ab \in H$ or $ab \in D$

Let $ab = h \in H$ then $b = a^{-1}h \in H$

Let $ab = d \in D$ then $a = d^{-1}b \in D$

$\Rightarrow\!\!\!/-$

Let $ab = h \in H$

then $b = a^{-1}h \in H$ $\quad\Rightarrow\Leftarrow$ $\qquad$ then $a = d^{-1}b \in D$ $\quad\Rightarrow\Leftarrow$

But $ab \in H \cup D$ $\quad\Rightarrow\Leftarrow$

So $H \cup D$ is not group.

Q) Index of $SL[2, \mathbb{R}]$ in $GL[2, \mathbb{R}]$

Ans:-